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COMPLETE CONVERGENCE AND THE LAW OF LARGE NUMBERS

By P. L. Hsu and Herbert Robbins

Department of Mathematical Statistics, University of North Carolina

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1. We begin by listing some standard definitions in the theory of probability. A *probability space* is a set Ω of elements ω together with a σ-field m of subsets of Ω on which is defined a completely additive measure P such that $P(\Omega) = 1$. A real-valued P-measurable function $X = X(\omega)$ is a *random variable*, and the function $F(x) = P\{X \leq x\}$, where $\{\ \}$ denotes the set of all ω such that the relation within the braces holds, is the *distribution function* of X. The sets of a sequence $A_1, A_2, \ldots$ are *independent* if for every finite set $i_1, \ldots, i_n$ of distinct integers, $P(\prod_{r=1}^{n} A_{i_r}) = \prod_{r=1}^{n} P(A_{i_r})$, and the random variables of a sequence $X_1, X_2, \ldots$ are independent if, for every sequence $x_1, x_2, \ldots$ of real numbers, the sets $\{X_1 \leq x_1\}, \{X_2 \leq x_2\}, \ldots$ are independent.

For purposes of comparison we list the following modes in which a sequence

$$X_1, X_2, \ldots \tag{1}$$

of random variables defined on Ω may converge to 0.

(*i*) The sequence (1) converges to 0 *in probability* if for every $\epsilon > 0$,

$$\lim_{n\to\infty} P\{|X_n| > \epsilon\} = 0.$$

(*ii*) The sequence (1) converges to 0 *with probability* 1 if for every $\epsilon > 0$,

$$\lim_{n\to\infty} P\{\{|X_n| > \epsilon\} + \{|X_{n+1}| > \epsilon\} + \ldots\} 0.$$

It is easily seen that this is equivalent to the usual condition, $P\{\lim_{n\to\infty} X_n = 0\} = 1$, and that (*ii*) implies (*i*) but not conversely.

2. We shall be concerned with a third mode of convergence, which, for want of a better name, we call *complete*.

(*iii*) The sequence (1) converges to 0 *completely* if for every $\epsilon > 0$,

$$\lim_{n\to\infty} [P\{|X_n| > \epsilon\} + P\{|X_{n+1}| > \epsilon\} + \ldots] = 0.$$

Clearly, (*iii*) implies (*ii*). The example: Ω = unit interval $0 < \omega < 1$, P = Lebesgue measure, $X_n = 1$ for $0 < \omega < \frac{1}{n}$ and 0 otherwise, shows that (*ii*) does not imply (*iii*).

Let us call two sequences of random variables $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$, defined, respectively, on probability spaces Ω and Ω_1, *F-equivalent*, if for every n the distribution function of Y_n is identical with that of X_n. Definitions (*i*) and (*iii*) are invariant under F-equivalence, while (*ii*) is not. However, *a sequence $X_1, X_2, \ldots$ of random variables converges to 0 completely if and only if every F-equivalent sequence converges to 0 with probability* 1. The necessity is obvious; to prove sufficiently consider a sequence $Y_1, Y_2, \ldots$ of independent random variables F-equivalent to the given sequence. If the sequence $Y_1, Y_2, \ldots$ converges to 0 with probability 1 then for any $\epsilon > 0$, $P\{\limsup_{n\to\infty}\{|Y_n| > \epsilon\}\} = 0$. Since the sets $\{|Y| > \epsilon\}$ are independent, it follows from a theorem of Borel-Cantelli[1] that $\sum_{n=1}^{\infty} P\{|Y_n| > \epsilon\} = \sum_{n=1}^{\infty} P\{|X_n| > \epsilon\} < \infty$.

It follows from this proof that if $X_1, X_2, \ldots$ is a sequence of *independent* random variables, then definitions (*ii*) and (*iii*) are equivalent.

3. Let the random variables X_n in (1) be independent with the same distribution function $F(x) = P\{X_n \leq x\}$ and such that the expectation $E(X_n) = \int_{-\infty}^{\infty} x dF(x) = 0$. The *strong law of large numbers* for identically distributed random variables states that the sequence of random variables $Y_1, Y_2, \ldots$, where for each n

$$Y_n = (X_1 + \ldots + X_n)/n \tag{2}$$

converges to 0 with probability 1. We shall show in Theorems 1 and 2 that under the same hypotheses the sequence (2) need not converge to 0 completely, but that it will do so under the further hypothesis that $\int_{-\infty}^{\infty} x^2 d F(x) < \infty$.

4. THEOREM 1. *Let (1) be a sequence of independent random variables with the same distribution function $F(x)$ and such that*

$$\int_{-\infty}^{\infty} x dF(x) = 0, \quad \sigma^2 = \int_{-\infty}^{\infty} x^2 dF(x) < \infty. \tag{3}$$

Then the sequence (2) converges to 0 completely; i.e., the series

$$\sum_{n=1}^{\infty} P\{|Y_n| > \epsilon\} \tag{4}$$

converges for every $\epsilon > 0$.

Proof. We shall prove the theorem for $\epsilon = 2$. This is no restriction since we can always consider $\frac{2}{\epsilon}X_n$ instead of X_n. Moreover, we may assume that $\sigma^2 > 0$.

Let $f(t) = \int_{-\infty}^{\infty} e^{itx}dF(x)$ be the characteristic function of the distribution $F(x)$. From (3) it follows that constants α, α', α'' exist such that

$$|1 - f(t)| \leq \alpha t^2, \quad |f'(t)| \leq \alpha' t, \quad |f''(t)| \leq \alpha''. \tag{5}$$

Choose and fix a positive δ so small that for $|t| \leq 4\delta$ the following conditions (6) and (7) are satisfied:

$$|\sin {}^1/_2 t| \geq Bt, \; |(1 - f(t))^2 - 4(1 - f(t)) \sin^2 {}^1/_2 t + 4 \sin^2 {}^1/_2 t| \geq Ct^2, \tag{6}$$

where B and C are constants,

$$|f(\,)| \neq 1 \qquad \text{except at } t = 0. \tag{7}$$

Let Z be a random variable distributed with the density $3(2\pi)^{-1}x^{-4}\sin^4 x$ and hence the characteristic function

$$\varphi(t) = \begin{cases} 1 - {}^3/_8 t^2 + {}^3/_{32}|t|^3, & |t| \leq 2, \\ {}^1/_{32}(4 - |t|)^2, & 2 \leq |t| \leq 4, \\ 0 & 4 \leq |t|. \end{cases} \tag{8}$$

We regard Z as independent of Y_n and use addition in this sense. Since

$$P\{|Y_n| > 2\} \leq P\left\{\left|Y_n + \frac{Z}{n\delta}\right| > 1\right\} + P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\} =$$

$$P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} - P\left\{\left|Y_n + \frac{Z}{n\delta}\right| \leq 1\right\} + 2P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\},$$

and since

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{Z}{n\delta}\right| > 1\right\} \leq \frac{3}{\pi}\sum_{n=1}^{\infty}\int_{n\delta}^{\infty}\frac{dx}{x^4} = \frac{1}{\pi\delta^3}\sum_{n=1}^{\infty}\frac{1}{n^3} < \infty,$$

it is sufficient to prove that

$$\sum_{n=1}^{N-1}\left[P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} - P\left\{\left|Y_n + \frac{Z}{n\delta}\right| \leq 1\right\}\right] = 0(1), \tag{9}$$

where $0(1)$ always denotes a quantity bounded with respect to N.

The characteristic function $f^n\left(\frac{t}{n}\right)\varphi\left(\frac{t}{n\delta}\right)$ of $Y_n + \frac{Z}{n\delta}$ vanishes for $|t| > 4n\delta$; hence by a well-known inversion formula,[2]

$$P\left\{\left| Y_n + \frac{Z}{n\delta}\right| \leq 1\right\} = \frac{1}{\pi}\int_{-4n\delta}^{4n\delta} f^n\left(\frac{t}{n}\right)\varphi\left(\frac{t}{n\delta}\right)\frac{\sin t}{t}dt =$$

$$\frac{1}{\pi}\int_{-4\delta}^{4\delta} f^n(t)\,\varphi\left(\frac{t}{\delta}\right)\frac{\sin nt}{t}dt.$$

Also,

$$P\left\{\left|\frac{Z}{n\delta}\right| \leq 1\right\} = \frac{1}{\pi}\int_{-4\delta}^{4\delta}\varphi\left(\frac{t}{\delta}\right)\frac{\sin nt}{t}dt.$$

Hence, by subtraction the left side of (9) is equal to

$$\frac{1}{\pi}\int_{-4\delta}^{4\delta}\frac{1}{t}\varphi\left(\frac{t}{\delta}\right)\sum_{n=1}^{N-1}(1 - f^n(t))\sin nt\,dt = \frac{1}{\pi}A_N, \text{ say.} \qquad (10)$$

From now on we write f for $f(t)$. Direct computation gives the result

$$\sum_{n=1}^{N-1}(1 - f^n)\sin nt = \frac{(1-f)^2\sin {}^1\!/_2Nt\sin {}^1\!/_2(N-1)t}{q(t)\sin {}^1\!/_2t}$$

$$+\frac{(1-f)\sin t}{q(t)} - \frac{4(1-f)\sin {}^1\!/_2t\sin {}^1\!/_2Nt\sin {}^1\!/_2(N-1)t}{q(t)}$$

$$+\frac{(1-f)f^N\sin Nt}{q(t)} - \frac{2(1-f^{N+1})\sin {}^1\!/_2t\cos(N - {}^1\!/_2)t}{q(t)}, \qquad (11)$$

where

$$q(t) = (f - e^{it})(f - e^{-it}) = (1-f)^2 - 4(1-f)\sin^2 {}^1\!/_2t + 4\sin^2 {}^1\!/_2t. \qquad (12)$$

By (6) we have $|q(t)| \geq Ct^2$, $|q(t)\sin {}^1\!/_2t| \geq C'|t|^3$, where C and C' are constants. Hence when (11) is substituted into (10) and the first inequality of (5) is used, we see that the first three terms merely contribute 0(1). Consequently,

$$A_N = \int_{-4\delta}^{4\delta}\varphi\left(\frac{t}{\delta}\right)\frac{(1-f)f^N\sin Nt - 2(1-f^{N+1})\sin {}^1\!/_2t\cos(N - {}^1\!/_2)t}{tq(t)}dt + 0(1). \qquad (13)$$

For $\delta \leq |t| \leq 4\delta$ we have, by (7), $|f| \neq 1$, hence $q(t) = |f - e^{it}|^2 \geq (1 - |f|)^2 \geq a > 0$. Therefore the part of the integral in (13) extended over the range $\delta \leq |t| \leq 4\delta$ is 0(1), so that (13) holds with 4δ replaced by δ in the two limits of integration. For $|t| \leq \delta$, however, $\varphi\left(\frac{t}{\delta}\right) = 1 - \frac{3t^2}{8\delta^2} + \frac{3|t|^3}{32\delta^3}$, and the terms with t^2 and $|t|^3$ are easily seen to contribute 0(1). Hence,

$$A_N = \int_{-\delta}^{\delta} \frac{(1-f)f^N \sin Nt - 2(1-f^{N+1}) \sin {}^1/_2 t \cos (N-{}^1/_2)t}{tq(t)} dt + 0(1). \tag{14}$$

Since

$$\left| \frac{1}{tq(t)} - \frac{1}{t^3} \right| = \left| \frac{t^2 - q(t)}{t^3 q(t)} \right| \leq k \frac{t^4}{|t|^5} = \frac{k}{|t|},$$

where k is a constant, the replacement of $tq(t)$ by t^3 in (14) will make a difference of only 0(1), so that

$$A_N = \int_{-\delta}^{\delta} \frac{(1-f)f^N \sin Nt}{t^3} dt - \int_{-\delta}^{\delta} \frac{2(1-f^{N+1}) \sin {}^1/_2 t \cos (N-{}^1/_2)t}{t^3} dt + 0(1). \tag{15}$$

Let the two integrals in (15) be denoted by I_N and J_N, respectively. We have

$$I_N = \frac{2}{N} \int_{-\delta}^{\delta} \frac{(1-f)f^N}{t^3} d(\sin^2 {}^1/_2 Nt) = 0(1) + \frac{2}{N} \int_{-\delta}^{\delta} \left\{ \frac{3(1-f)f^N}{t^4} + \frac{f^N f'}{t^3} - \frac{Nf^{N-1}(1-f)f'}{t^3} \right\} \sin^2 {}^1/_2 Nt dt.$$

Using the first two inequalities of (5) we obtain the result

$$|I_N| \leq 0(1) + (6\alpha + 2\alpha') \int_{-\infty}^{\infty} \frac{\sin^2 {}^1/_2 Nt}{Nt^2} dt + 2 \int_{-\infty}^{\infty} \frac{|1-f|\,|f'|}{|t^3|} dt = 0(1),$$

since the integral involving N is independent of N.

To deal with J_N we observe first that in J_N, $\sin {}^1/_2 t$ may be replaced by ${}^1/_2 t$ and f^{N+1} by f^N, the difference thus made being 0(1). Hence

$$J_N = \int_{-\delta}^{\delta} \frac{(1-f^N) \cos (N-{}^1/_2)t}{t^2} dt + 0(1) = \int_{-\infty}^{\infty} \frac{(1-f^N) \cos (N-{}^1/_2)t}{t^2} dt + 0(1).$$

We may replace $\cos (N-{}^1/_2)t$ by $\cos Nt$, since

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1-f^N}{t^2} (\cos (N-{}^1/_2)t - \cos Nt) dt = \frac{1}{\pi} \int_{-\infty}^{\infty} (1-f^N) \frac{\sin {}^1/_4 t}{{}^1/_4 t} \frac{\sin (N-{}^1/_4)t}{t} dt = P\{|U| \leq N - {}^1/_4\} - P\{|U + NY_N| \leq N - {}^1/_4\} = 0(1),$$

where U is a random variable independent of Y_N and whose characteristic function is $4/t \sin {}^1/_4 t$. Hence

$$J_N = 0(1) + \frac{1}{N}\int_{-\infty}^{\infty} \frac{1-f^N}{t^2}\, d \sin Nt = 0(1) + \frac{2}{N^2}\int_{-\infty}^{\infty} \left\{\frac{2(1-f^N)}{t^2} + \frac{Nf^{N-1}f'}{t^2}\right\} d \sin^2 {}^1/_2 t$$

$$= 0(1) + \frac{2}{N^2}\int_{-\infty}^{\infty} \left\{\frac{6(1-f^N)}{t^4} + \frac{4Nf^{N-1}f'}{t^3} - \frac{N(N-1)f^{N-2}f'^2}{t^2} - \frac{Nf^{N-1}f''}{t^2}\right\} \sin^2 {}^1/_2 Nt dt.$$

Using all the inequalities (5) we have

$$|J_N| \leq 0(1) + (12\alpha + 8\alpha' + 2\alpha'')\int_{-\infty}^{\infty} \frac{\sin^2 {}^1/_2 Nt}{Nt^2} dt + 2\int_{-\infty}^{\infty} \frac{|f'|^2}{t^2} dt = 0(1)$$

The proof is now complete.

5. By following the essential steps of the proof of Theorem 1 we obtain the following theorem, the proof of which is omitted from the present communication.

THEOREM 2. *If instead of conditions (3) we have*

$$\int_{-\infty}^{\infty} x dF(x) = 0, \quad \int_{-\infty}^{\infty} |x|^a dF(x) < \infty, \quad \int_{-\infty}^{\infty} x^2 dF(x) = \infty \qquad (16)$$

where a is some constant such that $\frac{1}{2}(1 + 5^{1/2}) \leq a < 2$, then the series (4) diverges for every $\epsilon > 0$. (Example: Let X_n be distributed with the density $|x|^{-3}$ for $|x| \geq 1$ and 0 elsewhere.)

Since the finiteness of the second integral in (16) would seem rather to favor than to oppose the convergence of (4), it may be conjectured that given the first condition of (3), the finiteness of σ^2 is not only sufficient but also necessary for the convergence of (4). We have not been able to prove this.

6. The following generalization[3] of the strong law of large numbers is an immediate consequence of Theorem 1 and the remarks in section 2.

THEOREM 3. *Let $X_n^{(r)}$ $(n = 1, 2, \ldots; r = 1 \quad \ldots, n)$ be an array of random variables with the same distribution function $F(x)$ and such that (1) $\int_{-\infty}^{\infty} x dF(x) = 0$, $\int_{-\infty}^{\infty} x^2 dF(x) < \infty$, and (2) for each n the random variables $X_1^{(n)}, \ldots X_n^{(n)}$ are independent. Then the sequence of random variables $Y_1, Y_2, \ldots$, where for each n, $Y_n = (X_1^{(n)} + \ldots + X_n^{(n)})/n$, converges to 0 with*

probability 1. (Note that we do not assume any relation of dependence or independence between $X_n^{(r)}$ *and* $X_m^{(s)}$ for $m \neq n$.)

[1] See M. Fréchet, *Recherches theoriques modernes*, Vol. 1, Paris, 1937, p. 27.

[2] See H. Cramér, *Mathematical methods of statistics*, Princeton, 1946, p. 93.

[3] Compare F. P. Cantelli, Considerazioni sulla legge uniforme dei grandi numeri ecc., *Giornale dell 'Istituto Italiano degli Attuari*, IV (1933), pp. 331–332; also H. Cramér, Su un theorema relativo alla leggi uniforme dei grandi numeri, *Ibid.*, V (1934), pp. 1–13.

GREEN'S FUNCTIONS FOR LINEAR DIFFERENTIAL SYSTEMS OF INFINITE ORDER

BY D. V. WIDDER

In these PROCEEDINGS[1] the author sketched a theory whereby a special differential equation of infinite order

$$\frac{\sin \pi D}{\pi} y(x) = \varphi(x) \tag{1}$$

could be solved by use of a Green's function. The operator on the left of this equation is interpreted to mean

$$\lim_{n \to \infty} D\left(1 - \frac{D^2}{1^2}\right) \cdots \left(1 - \frac{D^2}{n^2}\right) y(x),$$

where D is the operation of differentiation with respect to x. In preparing the details of this theory the author discovered that much more general differential equations could be treated by the same method. It is the purpose of the present note to outline this more general theory.

We define an entire function $E(s)$ as follows:

$$E(s) = s \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right), \tag{2}$$

where the constants a_k are real and such that

$$0 < a_1 < a_2 < \cdots. \tag{3}$$

$$\sum_{k=1}^{\infty} \frac{1}{a_k^2} < \infty. \tag{4}$$

Consider now the differential system

$$E(D)y(x) = \varphi(x) \tag{5}$$

$$y(-\infty) = 0 \tag{6}$$

$$y(+\infty) = \int_{-\infty}^{\infty} \varphi(t)dt. \tag{7}$$

We are assuming that $\varphi(x)$ is a given continuous function which is absolutely integrable on $(-\infty, \infty)$. In particular, if $a_k = k$, equation (5) reduces to equation (1).

It is natural to define the Green's function, $G(x)$, for the system (5) (6) (7) as the limit of the Green's function, $G_{2n+1}(x)$, of the "truncated" system

$$D \prod_{k=1}^{n} \left(1 - \frac{D^2}{a_k^2}\right) y(x) = \varphi(x) \tag{8}$$

with boundary conditions (6) and (7). We recall the definition[2] of $G_{2n+1}(x)$. It satisfies the semihomogeneous system

$$D \prod_{k=1}^{n} \left(1 - \frac{D^2}{a_k^2}\right) y(x) = 0$$

$$y(-\infty) = 0, \; y(+\infty) = 1$$

for all x different from zero. It is continuous with its first $(2n-1)$ derivatives for all x. Its $2n$th derivative is continuous except at $x = 0$, where

$$y^{(2n)}(0+) - y^{(2n)}(0-) = (-1)^n a_1^2 a_2^2 \ldots a_n^2.$$

It is easy to see that these conditions determine $G_{2n+1}(x)$ uniquely and that the unique solution of the system (8) (6) (7) is

$$y(x) = \int_{-\infty}^{\infty} G_{2n+1}(x - t)\varphi(t)dt.$$

We can, in fact, write down an explicit integral formula for $G_{2n+1}(x)$:

$$G_{2n+1}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}\, ds}{s \prod_{k=1}^{n} \left(1 - \frac{s^2}{a_k^2}\right)} \qquad 0 < c < a_1.$$

This integral can in turn be evaluated by use of the calculus of residues, and is two distinct linear combinations of exponential functions in the two intervals $(-\infty, 0)$ and $(0, \infty)$.

We are next able to show that $G_{2n+1}(x)$ tends to a limit $G(x)$ as n becomes infinite, and that this function yields a solution

$$y(x) = \int_{-\infty}^{\infty} G(x - t)\varphi(t)dt \tag{9}$$

of the system (5) (6) (7). In the course of the proof of these facts, we show that the reciprocal of the function $E(s)$ is a bilateral Laplace transform.

$$\frac{1}{E(s)} = \int_{-\infty}^{\infty} e^{-st} G(t)dt,$$

convergent in the interval $0 < s < a_1$. In fact the determining function, as we have indicated by the notation, is the Green's function described above.

We summarize the above results in the following theorem.

THEOREM. *If* $\varphi(x) \epsilon C \cdot L$ *in* $(-\infty < x < \infty)$, *if* $E(s)$ *is an entire function defined by* (2) (3) (4), *and if*

$$G(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{E(s)} ds \qquad 0 < c < a_1,$$

then the function $y(x)$ *defined by* (9) *satisfies the system* (5) (6) (7).

We have thus obtained a real linear differential inversion formula for the general Faltung-equation (9). In particular, if $a_k = k$, $G(x)$ becomes

$$G(x) = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{\sin \pi s} ds = \frac{1}{1 + e^{-x}},$$

and equation (9) becomes

$$y(x) = \int_{-\infty}^{\infty} \frac{\varphi(t) e^{-t}}{e^{-t} + e^{-x}} dt,$$

or

$$f(x) = y\left(\log \frac{1}{x}\right) = \int_0^{\infty} \frac{\varphi\left(\log \frac{1}{t}\right)}{x + t} dt \qquad (10)$$

after an exponential change of variable. The same exponential change of variable applied to the differential operator (1) shows that

$$\lim_{k \to \infty} \frac{(-1)^{k-1}}{k!(k-2)!} [x^{2k-1} f^{(k-1)}(x)]^k = \varphi\left(\log \frac{1}{x}\right).$$

We thus rediscover, as a special case of the present theory, the author's real inversion formula for the Stieltjes transform.[3] In the light of this general theory the original discovery of that formula seems somewhat fortuitous, depending as it did on the Laplace asymptotic evaluation of an integral of the form

$$\int_0^{\infty} [g(t)]^n \varphi(t) dt. \qquad (11)$$

It was the special nature of the constants a_k as integers that introduced powers of a function $g(t)$. We are now able to retain the essential features of the method even though the integral is now replaced by another which no longer involves powers of a function. One of the outstanding features of the present theory is that it enables one to set up a whole new class of inversion formulas for integral transforms. One may either discover the

kernel of the transform from a given inversion operator, or, conversely, set up the inversion operator when the kernel is given. The present approach is related to the fundamental work of N. Wiener on the operational calculus.[4] It should be observed, however, that there is here no appeal to the L^2-Fourier transform theory.

[1] Widder, D. V., "The Green's Function for a Differential System of Infinite Order," *Proc. Nat. Acad. Sci.*, **26**, 213–215 (1940).

[2] See, for example, M. Bôcher, *Leçons sur les méthodes de Sturm*, Paris (1917), Chap. V.

[3] Widder, D. V., "The Stieltjes Transform," *Trans. Am. Math. Soc.*, **43**, 7–60 (1938).

[4] Wiener, N., "The Operational Calculus," *Math. Annalen*, **95**, 557–584 (1926).

SPECIAL VALUES OF $e^{k\pi}$, COSH $(k\pi)$ AND SINH $(k\pi)$ TO 136 FIGURES

BY HORACE S. UHLER

YALE UNIVERSITY, NEW HAVEN

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The present investigation was undertaken with two primary objects in view, namely, to obtain certain heretofore uncomputed values of $e^{k\pi}$, cosh $(k\pi)$ and sinh $(k\pi)$, and to extend appreciably the degree of approximation of data previously found by others and by the author.

The actual calculations were based on his 137-place table of logarithms because preliminary tests of available infinite series showed that their application would have necessitated a much greater expenditure of labor and time than the radix logarithms. For example, the 79th term of the series $e^z = \sum_{n=0}^{\infty} (z^n/n!)$, when $z = \pi/6$, has 139 zeros between the decimal point and the leading figures 70673. Seventy-nine terms would not have been deterrent but for $n > 2$ examination of page 84 of the invaluable *Index of Mathematical Tables*[1] revealed the facts that the published values of π^n required for the earlier terms of the series attained 41 figures in one dubious case and that for somewhat lower degrees of approximation only 32 consecutive entries were tabulated. Again, consider the series $e^{i\sin^{-1}x}$ which was chosen by the author in an earlier article[2] because x has the rational value 1/2 when $\sin^{-1} x = \pi/6$. The general expression for the terms involving even powers of x may be written

$$t_{2k+1} = (0^2 + 1)(2^2 + 1)(4^2 + 1)\ldots[(2k - 2)^2 + 1]x^{2k}/(2k!).$$

When $x = 1/2$ the course of this function of k and the prohibitively slow convergence of the series may be seen from the following values of t_{2k+1} $t_{83} = 5.0977 \times 10^{-28}$, $t_{163} = 1.5209 \times 10^{-52}$, $t_{243} = 6.8957 \times 10^{-77}$, $t_{323} =$

3.7173×10^{-101}, $t_{403} = 1.7512 \times 10^{-125}$ and $t_{451} = 6.6138 \times 10^{-140}$, where $k = 41, 81, 121, 161, 201$ and 225, respectively.

A condensed account of the details of the performance of the calculations for table 1 will be presented now in order to emphasize the precautions taken by the author to reduce the probability of error to an extremely small value.

First $e^{+\pi/2}$ was calculated from the original master strips upon which the 137-place printed table of negative radix logarithms had been based. All of these fundamental data extended well beyond the 137th place of decimals. The most error-sensitive part of the work consisted as usual in multiplying together mentally the binomial factors as obtained from the logarithms. Hence, this set of operations was performed three times. Next $e^{-\pi/2}$ was computed from $e^{+\pi/2}$ by machine multiplication reversed. The checking product $e^{+\pi/2} \cdot e^{-\pi/2}$ gave 150 consecutive 9's. Then $e^{+\pi}$ and $e^{-\pi}$ were found by hand-machine squaring, respectively, $e^{+\pi/2}$ and $e^{-\pi/2}$. $e^{+\pi/2} \cdot e^{-\pi/2}$ was performed twice. The check $e^{+\pi} \cdot e^{-\pi}$ yielded 150 successive 9's.

After this $e^{-\pi/192}$ was computed from the master data of negative radix logarithms with the same extreme care as explained above. By repeated squaring the values of $e^{-\pi/96}$, $e^{-\pi/48}$, $e^{-\pi/24}$, $e^{-\pi/12}$, $e^{-\pi/6}$ and $e^{-\pi/3}$ were calculated. In this sequence, and for all machine multiplications referred to below, the act of multiplying was performed independently twice, first with the figures segregated as octads and second with nonad grouping. To check the work $e^{-\pi/2}$ was obtained from $e^{-\pi/3} \cdot e^{-\pi/6}$. This approximation to $e^{-\pi/2}$ was less than that derived at the beginning from a different set of logarithms by 3.547 units in the 140th decimal place. This means a per cent error of -1.706×10^{-137} for the second determination of $e^{-\pi/2}$ when the first value is taken as standard.

Diminishing in details, reciprocating $e^{-\pi/192}$ gave $e^{+\pi/192}$. The product of these numbers produced 151 consecutive 9's. Squaring led to $e^{+\pi/96}$, $e^{+\pi/48}$, $e^{+\pi/24}$, $e^{+\pi/12}$, $e^{+\pi/6}$ and $e^{+\pi/3}$. The product $e^{+\pi/3} \cdot e^{+\pi/6}$ was greater than the value of $e^{+\pi/2}$ as first found by 8.206 units in the 139th decimal place. This is equivalent to a per cent error of $+ 1.706 \times 10^{-137}$.

$e^{+\pi/192} \cdot e^{+\pi/96}$ gave $e^{+\pi/64}$. Successive squaring produced $e^{+\pi/32}$, $e^{+\pi/16}$, $e^{+\pi/8}$, $e^{+\pi/4}$ and $e^{+\pi/2}$. The "error" equaled $+7.381$ units in the 139th decimal place or $+1.534 \times 10^{-137}$ per cent.

Finally $e^{-\pi/192} \cdot e^{-\pi/96}$ gave $e^{-\pi/64}$. Repeated squaring furnished $e^{-\pi/32}$, $e^{-\pi/16}$, $e^{-\pi/8}$, $e^{-\pi/4}$ and $e^{-\pi/2}$. The "error" was -3.546 units in the 140th decimal place, that is -1.706×10^{-137} per cent.

It should now be clear that the interval between the reciprocal pair $(e^{+\pi/192}, e^{-\pi/192})$ and the pair $(e^{+\pi/2}, e^{-\pi/2})$ had been bridged successfully by four different sequences so that the constants in table 1 are bound together as an unimpeachable unit. Nevertheless, as an extreme precaution

the product $e^{-\pi/16} \cdot e^{+\pi/16}$ was formed and found to equal $1 + 2.078 \times 10^{-144}$. The author considered the gain in confidence which might attach to multiplying together the members of the ten remaining reciprocal pairs having the respective values of $k = 1/96, \cdots, \pm 1/24, \pm 1/12, \cdots, \pm 1/3$ to be negligible. He felt justified in guaranteeing the accuracy of 136 overall figures for all of the 56 numbers in tables 1, 2 and 3, and in supposing that 138 figures may well be reliable if the last digits be rounded off conventionally. He also deemed it superfluous to present a full set of complex equivalents such as $e^{-\pi/2} = i^i$, $e^{-\pi/4} = i^{i/2} = (1 + i)^i/2^{i/2}$, $e^{-\pi/6} = i^{i/3} = (\sqrt{3} + i)^i/2^i$, etc.

It is now appropriate to compare the short table[3] published for the author in the year 1921 with the new table 1. In terms of unity in the 53rd decimal place the errors are $+0.04$, -0.6, -0.5, $+0.5$, -1.6, $+0.6$, -3.9, $+0.6$, -0.4 and $+2.4$, respectively, for $\cosh(\pi/6)$ or σ_1, $\sinh(\pi/6)$ or σ_2, $e^{+\pi/6}$, $e^{-\pi/6}$, $e^{+\pi/3}$, $e^{-\pi/3}$, $e^{+\pi/2}$, $e^{-\pi/2}$, $e^{+\pi}$ and $e^{-\pi}$. Hence, the chief comment on accuracy by the author in his earlier paper, to wit: "It is concluded that the calculated values of $e^{\pi/6}$ and $e^{-(\pi/6)}$ given above are certainly correct to 52 decimal places and they are probably in error by not more than two units in the fifty-third place." was too cautious since the estimated two units have just been shown to be a half-unit for each of these two constants.

With regard to the entries in tables 2 and 3 the only particular remark to be made is that the relation $\cosh^2 x - \sinh^2 x = 1$ was not employed since it does not afford a significant check. Any pair of numbers c and s which are computed from $c = (m + n)/2$ and $s = (m - n)/2$ will automatically satisfy the condition $c^2 - s^2 = 1$ when m and n are mutually reciprocal ($mn = 1$) regardless of whether m and n are correct or false. The data recorded in tables 2 and 3 were finally tested, respectively, by comparing $\cosh(k\pi) + \sinh(k\pi)$ with $e^{+k\pi}$ and $\cosh(k\pi) - \sinh(k\pi)$ with $e^{-k\pi}$

TABLE 1

$e^{-\pi}$

0.043213 91826 37722 49774 41773 71717 28011 27572 81098 10633
08298 07196 87401 05076 57570 17967 69813 99599 61901 08438
70168 06964 59766 20563 26519 83177 96586 643

$e^{-\pi/2}$

0.20787 95763 50761 90854 69556 19834 97877 00338 77841 63176
96080 75135 88305 54198 77285 48213 97886 00277 86542 60353
40521 77330 72350 21808 19061 97303 74663 987

$e^{-\pi/3}$

0.35091 98071 78410 96756 57367 15996 95305 83625 73153 62096
17406 52384 89783 45471 91387 90591 77985 90909 88323 39147
06666 33285 85352 68847 09190 14152 65579 259

$e^{-\pi/4}$

0.45593 81277 65996 23676 59212 94728 02941 94166 04365 23785
18699 96290 97942 91596 45224 44006 65252 86099 31970 89010
89717 15729 22467 00469 21645 96570 72613 676

$e^{-\pi/6}$

0.59238 48471 88388 98366 54163 32661 91948 74141 14145 73545
87885 86614 33131 34832 38837 31793 49841 44951 21797 99141
76059 55032 23110 94028 40466 25757 73779 619

$e^{-\pi/8}$

0.67523 19066 55777 21703 20677 47365 67745 52428 58398 76998
90153 62810 86630 75497 42403 88968 02189 34997 39154 35601
82505 66947 52247 18507 66691 11544 94852 116

$e^{-\pi/12}$

0.76966 54124 93239 80757 37687 07230 48101 92320 94381 45579
43945 37330 57085 91181 65585 87353 79624 51895 66972 17517
05126 94148 36376 35938 19214 09114 54172 251

$e^{-\pi/16}$

0.82172 49580 33877 18268 07042 67874 56518 14786 26393 88136
50546 82104 78626 67328 29717 63362 32248 04748 72616 77689
30630 20435 02711 43458 19195 75361 79638 557

$e^{-\pi/24}$

0.87730 57690 98345 66958 38146 46330 33330 36446 41138 05807
96653 77824 91012 19286 58909 33398 97527 78407 10063 25151
10456 22895 36087 95217 57535 38395 83850 623

$e^{-\pi/32}$

0.90649 04621 85828 61749 19356 14428 78207 71753 10703 01884
27954 51723 37437 17268 52109 56926 31641 32136 87064 04019
48480 80078 74342 73205 24724 88735 18654 607

$e^{-\pi/48}$

0.93664 60212 36595 90634 60522 74229 76655 38448 83531 30346
03214 97622 82015 35203 78452 73519 13683 08268 43321 87011
69933 53289 27014 10931 93187 77821 07073 897

$e^{-\pi/64}$

0.95209 79267 83704 64850 57371 28524 78217 47368 50709 61530
20059 23646 99060 73544 68178 19136 92627 70109 16779 60607
15174 75247 57469 65910 64497 44635 78478 952

$e^{-\pi/96}$

0.96780 47433 42682 68575 15312 11040 92577 36808 13730 06534
88551 75905 15613 56933 08794 76890 34445 46481 63610 49965
58229 25442 11221 67359 42414 34417 10376 428

$e^{-\pi/192}$

0.98377 06761 95770 07076 45433 46905 86541 93981 56612 48425
61504 73315 01861 14284 86299 88069 84503 08483 02298 55201
10031 84251 03613 06973 17061 46654 75682 679

$e^{+\pi/192}$

1.01649 70599 31678 93392 79156 91977 04841 01851 69753 48016
45221 51728 77947 03160 59876 08158 41919 12811 78737 71911
40026 17671 74146 56594 21738 12419 24174 12

$e^{+\pi/96}$

1.03326 62728 49747 27440 79888 73714 29670 23989 95177 65525
96632 08760 80487 10875 33255 87611 11374 91682 69932 52813
67847 09374 25505 72658 50611 91924 40386 09

$e^{+\pi/64}$

1.05031 21284 78332 07288 13100 54743 31723 52734 41557 23629
70275 41810 21062 57144 78851 52191 38322 78172 47476 78070
74928 68061 37557 46854 30037 05370 40045 69

$e^{+\pi/48}$

1.06763 91906 08808 38146 14777 71484 30623 62679 69083 97978
68746 03773 54711 14283 14075 56843 00682 22767 51172 82414
77516 01271 92554 03224 88574 98282 14958 54

$e^{+\pi/32}$

1.10315 55672 28684 33894 60412 81997 43095 22145 66262 92396
02550 30971 16357 39579 78763 16667 05051 27890 54487 28637
21238 67293 93445 02833 93112 75787 09558 62

$e^{+\pi/24}$

1.13985 34413 23831 47486 81339 33638 76384 66648 24898 32557
63167 24018 88289 99447 14199 93899 39466 02696 23422 47749
77947 06700 11765 10426 61694 65093 26266 27

$e^{+\pi/16}$

1.21695 22055 07640 29224 03772 32573 75356 27507 21240 12904
30191 15215 39645 38850 39433 17871 43394 16971 10496 70703
18256 14080 33875 12990 53549 49519 84234 55

$e^{+\pi/12}$

1.29926 58676 97781 32296 99617 95129 29210 26875 68754 77964
06977 40006 55841 88751 53530 32172 97802 77089 46254 42062
48874 34933 84293 47909 62294 29885 28814 02

$e^{+\pi/8}$

1.48097 26704 89909 97123 52415 92730 75004 77577 57711 23219
60912 61162 58951 82273 93344 38056 13382 19034 61143 69902
18669 03919 22545 01536 79608 24540 28835 45

$e^{+\pi/6}$

1.68809 17949 64468 60061 68476 28096 78229 41196 81189 28718
23125 68405 60788 89433 93169 66280 20423 23480 95967 58606
00207 57323 50767 66184 69723 33214 55304 54

$e^{+\pi/4}$

2.19328 00507 38015 45655 97696 59278 73822 34616 37641 99427
23348 58015 91865 70268 64189 23693 41265 22812 57816 94047
11677 59357 90761 56946 47041 60085 07626 05

$e^{+\pi/3}$

2.84965 39082 26361 49747 41273 19852 90439 39640 06102 78112
68817 43238 77848 81424 49158 84145 05473 44562 77101 70814
93266 95773 11240 85932 40160 96305 55410 98

$e^{+\pi/2}$

4.81047 73809 65351 65547 30356 66703 83312 63901 70874 66453
49400 20815 48924 25519 04891 58213 67487 04766 58388 33546
57282 22273 56991 29262 20334 57206 46343 99

$e^{+\pi}$

23.14069 26327 79269 00572 90863 67948 54738 02661 06242 60021
19934 45046 40952 43423 50690 45278 35169 71997 06754 92196
75952 70480 10877 73144 42804 44146 93835 8

TABLE 2

sinh (π/192)

0.016363 19186 79544 31581 68617 25355 91495 39350 65704 97954
18583 92068 80429 44378 67881 00442 87080 21643 82195 83551
49971 67103 52667 48105 23383 28822 42457 185

sinh (π/96)

0.032730 76475 35322 94328 22883 13366 85464 35909 07237 94955
40401 64278 24367 69711 22305 53603 84647 26005 31610 14240
48089 19660 71420 26495 40987 87536 50048 297

sinh (π/64)

0.049107 10084 73137 12187 78646 31092 67530 26829 54238 10497
51080 90816 10009 18000 53366 65272 28475 40316 53485 87317
98769 64069 00439 04718 27698 03673 07833 702

sinh (π/48)

0.065496 58468 61062 37557 71274 86272 69841 21154 27763 38163
27655 30753 63478 95396 78114 16619 34995 72495 39254 77015
37912 39913 27699 61464 76936 02305 39423 205

sinh (π/32)

0.098332 55252 14278 60727 05283 37843 24437 51962 77799 52558
72978 96238 94601 11556 33267 98703 67049 78768 37116 23088
63789 36075 95511 48143 41939 35259 54520 049

sinh (π/24)

0.13127 38361 12742 90264 21596 43654 21527 15100 91880 13374
83256 73096 98638 90080 27645 30250 20969 12144 56679 61299
33745 41902 37838 57604 52079 63348 71207 823

sinh (π/16)

0.19761 36237 36881 55477 98364 82349 59419 06360 47423 12383
89822 16555 30509 35761 04857 77254 55573 06111 18939 96506
93812 96822 65581 84766 17176 87079 02297 995

sinh (π/12)

0.26480 02276 02270 75769 80965 43949 40554 17277 37186 66192
31516 01337 99377 98784 93972 22409 59089 12596 89641 12272
71873 70392 73958 55985 71540 10385 37320 884

$\sinh(\pi/8)$

0.40287 03819 17066 37710 15869 22682 53629 62574 49656 23110
35379 49175 86160 53388 25470 24544 05596 42018 60994 67150
18081 68485 85148 91514 56458 56497 66991 665

$\sinh(\pi/6)$

0.54785 34738 88039 80847 57156 47717 43140 33527 83521 77586
17619 90895 63828 77300 77166 17243 35290 89264 87084 79732
12074 01145 63828 36078 14628 53728 40762 462

$\sinh(\pi/4)$

0.86867 09614 86009 60989 69241 82275 35440 20225 16638 37821
02324 30862 46961 39336 09482 39843 38006 18356 62923 02518
10980 21814 34147 28238 62697 81757 17506 189

$\sinh(\pi/3)$

1.24936 70505 23975 26495 41953 01927 97566 78007 16474 58008
25705 45426 94032 67976 28885 46776 63743 76826 44389 15833
93300 31243 62944 08542 65485 41076 44915 86

$\sinh(\pi/2)$

2.30129 89023 07294 87346 30400 23434 42717 81781 46516 51638
26659 72839 80309 35660 13803 04999 84800 52244 35922 86596
58380 22471 42320 53727 00636 29951 35840 00

$\sinh(\pi)$

11.54873 93572 57748 37797 73343 15388 40968 44951 89066 39478
94552 32163 36106 16457 92466 71740 79094 16018 55282 40676
44467 94891 82450 55544 05076 22914 57088 6

TABLE 3

$\cosh(\pi/192)$

1.00013 38680 63724 50234 62295 19441 45691 47916 63182 98221
03363 12521 89904 08722 73087 98114 13211 10647 40518 13556
25029 00961 38879 81783 69399 79536 99928 40

$\cosh(\pi/96)$

1.00053 55080 96214 98007 97600 42377 61123 80399 04453 86030
42591 92332 98050 33904 21025 32250 72910 19082 16771 51389
63038 17408 18363 70008 96513 13170 75381 26

$\cosh(\pi/64)$

1.00120 50276 31018 36069 35235 91634 04970 50051 46133 42579
95167 32728 60061 65344 73514 85664 15475 24140 82128 19338
95051 71654 47513 56382 47267 25003 09262 32

$\cosh(\pi/48)$

1.00214 26059 22702 14390 37650 22857 03639 50564 26307 64162
35980 50698 18363 24743 46264 15181 07182 65517 97247 34713
23724 77280 59784 07078 40881 38051 61016 22

$\cosh(\pi/32)$

1.00482 30147 07256 47821 89884 48213 10651 46949 38482 97140
15252 41347 26897 28424 15436 36796 68346 30013 70775 66328
34859 73686 33893 88019 58918 82261 14106 61

$\cosh(\pi/24)$

1.00857 96052 11088 57222 59742 89984 54857 51547 33018 19182
79910 50921 89651 09366 86554 63649 18496 90551 66742 86450
44201 64797 73926 52822 09615 01744 55058 45

$\cosh(\pi/16)$

1.01933 85817 70758 73746 05407 50224 15937 21146 73817 00520
40368 98660 09136 03089 34575 40616 87821 10859 91556 74196
24443 17257 68293 28224 36372 62440 81936 55

$\cosh(\pi/12)$

1.03446 56400 95510 56527 18652 51179 88656 09598 31568 11771
75461 38668 56463 89966 59558 09763 38713 64492 56613 29789
77000 64541 10334 91923 90754 19499 91493 13

$\cosh(\pi/8)$

1.07810 22885 72843 59413 36546 70048 21375 15003 08055 00109
25533 11986 72791 28885 67874 13512 07785 77016 00149 02752
00587 35433 37396 10022 23149 68042 61843 78

$\cosh(\pi/6)$

1.14023 83210 76428 79214 11319 80379 35089 07668 97667 51132
05505 77509 96960 12133 16003 49036 85132 34216 08882 78873
88133 56177 86939 30106 55094 79486 14542 08

$\cosh(\pi/4)$

1.32460 90892 52005 84666 28454 77003 38382 14391 21003 61606
21024 27153 44904 30932 54706 83850 03259 04455 94893 91529
00697 37543 56614 28707 84343 78327 90119 86

$\cosh(\pi/3)$

1.60028 68577 02386 23251 99320 17924 92872 61632 89628 20104
43111 97811 83816 13448 20273 37368 41729 67736 32712 54980
99966 64529 48296 77389 74675 55229 10495 12

$\cosh(\pi/2)$

2.50917 84786 58056 78200 99956 43269 40594 82120 24358 14815
22740 47975 68614 89858 91088 53213 82686 52522 22465 46949
98901 99802 14670 75535 19698 27255 10503 99

$\cosh(\pi)$

11.59195 32755 21520 62775 17520 52560 13769 57709 17176 20542
25382 12883 04846 26965 58223 73537 56075 55978 51472 51520
31484 75588 28427 17600 37728 21232 36747 2(5)

[1] Fletcher, A., Miller, J. C. P., and Rosenhead, L., *An Index of Mathematical Tables*, London (1946).

[2] Uhler, H. S., *Amer. Math. Monthly*, **28**, 114 (1921).

[3] *Ibid.*, 115.

A REVERSE MUTATION TO A "REMOTE" ALLELE IN THE HOUSE MOUSE

By C. C. Little and K. P. Hummel

Roscoe B. Jackson Memorial Laboratory, Bar Harbor

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The gene A^w producing the white bellied agouti coat pattern in mice is one of a series of multiple alleles. Epistatic to it is the gene A^y for yellow coat color and hypostatic in descending order are: A = gray bellied agouti, a^t = black and tan, and a = non-agouti.

There have been a number of recorded cases of mutations from other alleles of the agouti series to A^w (Little, 1916[1]; Keeler, 1931[2]). In each case the mutation was or could have been a "single-step" reverse mutation from A.

The fact that the genotype Aa^t has synthetically a coat color practically identical with A^wA^w mice has led Pincus (1929)[3] and Keeler (*loc. cit.*) to raise the question as to whether there might be a dominant gene for white or light belly closely linked with the agouti locus. No crossovers have, however, been recorded.

As Gruneberg (1943, p. 30)[4] points out, a mutation from A^w to a or one in the reverse direction would provide strong evidence in favor of the hypothesis of multiple alleles and against that of linkage.

The occurrence of three A^wa individuals in different litters from a single mating of the closely inbred dilute brown (*dba*) strain of mice which has previously produced only *aa* animals[5] since 1909 has been observed and is here recorded. The three individuals occurred among the progeny of ♀1 × ♂10, there being 18 normal *aa* young.

The three mutant young differed in coat color from their *dba* parents only in the change from a to A^w. One of the two young is still too young to breed. One died in infancy. The other ♀2 has been crossed with non-agouti *aa* individuals and has produced 6 white-bellied agouti A^wa individuals and 5 non-agouti *aa* thus showing that the genetic formula of the mutant ♀2 was A^wabbdd as would be expected if the genetic change occurred in only one of the *dba* parents.

The frequency of the appearance of the mutant mice in the original mating suggests that one gonad of the mutating parent is a mosaic in respect to the formation of A^w and a gametes.

An effort has been made to determine which parent is the animal thus affected. Since the female produced three litters by a dilute brown male in order to provide the primary data on which the occurrence of the mutation is based there has not been time as yet to make a significant quantitative study of her progeny by unrelated males.

Breeding tests of the father of the mutants have, however, been made. When crossed with unrelated non-agouti females (*aa*) he has given a total of 43 young, all non-agouti. It, therefore, appears that he alone is not responsible for the appearance of the mutants. If one parent is responsible it is apparently the female.

There remains a theoretical possibility to be tested by further breeding. The ratio of non-mutant (*aa*) to mutant ($A^w a$) mice is 18 to 3. This approximates a theoretical 3:1 ratio of 15:5. If both parents carried a hypothetical gene (*m*) which when contributed by both to the embryo caused a mutation from *a* to A^w in one of the pair of chromosomes in which the agouti locus is situated and if this mutation occurred in the *mm* embryos at or soon after fertilization a high incidence of mutants comparable to that observed would be expected.

The *m* gene would be carried by two-thirds of the *aa* mice produced from the mating of the original parents and in 50% of the *aa* mice produced from any outcross of either parent. Inbreeding such *aa* mice should recreate combinations which were *mm* and which, therefore, mutated. It may be admitted that this explanation is somewhat improbable but the high relative incidence of the mutation among the original sibship is very unusual when compared with previous records of dominant mutation incidence in rodents and it seems desirable to record any possible explanation which might account for it.

The present case is interesting because of the following facts:

(1) It represents a reverse mutation which has apparently "skipped" two alleles (a^t and *A*) epistatic to *a* in order to reach the A^w allele. In such a multiple allelomorphic series it is evident that the configuration of the gene *a* can become that of its allele A^w without evidence of its having passed through the two genetically "interstatic" alleles, a^t and *A*.

(2) Its frequency of appearance is sufficient to suggest a very early division of the gonad of the mutating parent into "$A^w a$" and "*aa*" bearing cells.

(3) It makes the truly multiple allelomorphic series of A^y, A^w, *A*, a^t and *a* appear much more probable than the hypothesis of close linkage of a gene for "light belly" with the "agouti" locus.

[1] Little, C. C., *Am. Nat.*, **50**, 335–349 (1916).

[2] Keeler, C. E., *Proc. Nat. Acad. Sci.*, **17**, 497–499 (1931).

[3] Pincus, G., *Proc. Nat. Acad. Sci.*, **15**, 85–88 (1929).

[4] Gruneberg, H., The Genetics of the Mouse, Cambridge Univ. Press, pp. xii + 412 (1943).

[5] Other color mutations in the *dba* strain have been recorded as follows: (1) from *P* to *p*—dark eye to pink eye, Little, 1916; (2) from self coat *Bt* to white belted *bt*, Murray, 1936; (3) from normal pigmentation to a new type of dilution "misty," Woolley, 1945. Two other "dilute" mutants—one of them possibly identical with the "leaden" type of pigment production—are now being investigated.

CHROMOSOMES OF THE MINK*

By Richard M. Shackelford and Louise Wipf

Departments of Genetics and Veterinary Science, University of Wisconsin

Chromosomes of only one of the Mustelidae have been studied in so far as the writers are aware. Koller,[2] working with the testes of both oestrous and anoestrous individuals, found the diploid chromosome number of the ferret (*Putorius furo*, L.) to be 34. He observed an unequal pair which he considered the sex chromosomes, the *Y* chromosome being "3–4 times smaller" than the *X* chromosome. He stated further that the sex chromosomes were "precocious in their behavior during meiotic prophase," and that the "*X Y* bivalent lies off the metaphase plate."

The present report is concerned with the number and general morphology of the chromosomes of the standard dark color phase of ranch-bred mink (*Mustela vison*, Peale and Beauvois). Testes from six animals were removed in March and April (three animals in 1942, one in 1943 and two in 1946), and the seminiferous tubules were immediately teased out and placed in Carnoy's alcohol-acetic acid-chloroform solution (7:2:1). Since speedy fixation is necessary for the immediate arrest of division figures, the operation was completed in less than five minutes for each pair of testes. The tubules were allowed to remain in the fixative from 30 minutes to one hour, then stored in 80 per cent alcohol. Temporary mounts prepared by the aceto-carmine smear method were made from this material. Polar views of diploid equatorial plates show 28 to be the chromosome number of the mink (Fig. 1A). The chromosomes vary in length, the shortest being about one-third that of the longest. They are longitudinally split with median or subterminal spindle-attachment regions.

Mitotic as well as meiotic divisions are relatively abundant in the seminiferous tubules of the testes of mink during the breeding season. Homologous pairs of chromosomes are easily identified in several instances. In order to study their comparative morphology, each of the chromosomes has been arranged with what appears to be its counterpart using total length, position of the spindle-attachment region and relative lengths of the two arms as a guide (Fig. 1B). Thirteen pairs match as if they are homologous chromosomes. The two chromosomes of the fourteenth

FIGURE 1A. Mitotic equatorial plate showing the diploid chromosome number of 28.

FIGURE 1B. Chromosomes from 1A arranged as homologous pairs.

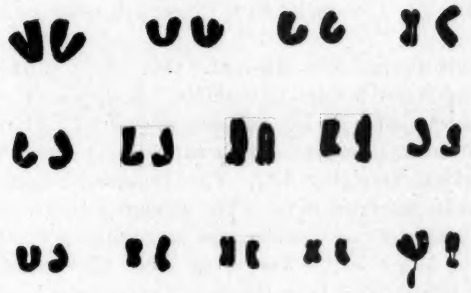
FIGURE 2. Late heterotypic prophase, showing the chromosome with the median attached satellite (*s*), and the nucleolus (*n*).

FIGURE 3. Polar view of a homeotypic equatorial plate.

FIGURE 4. Side view of heterotypic chromosomes approaching the equatorial plate. Two chromosomes show chiasmata (*c*). Note astral formations.

(*Remainder of legend at bottom of p. 45*)

1 A



1 B

2

3

5

4

6

FIGURE 5. Heterotypic chromosomes at anaphase.
FIGURE 6. Mitotic chromosomes at anaphase.
All figures × 2190.

pair are notably different. One has a subterminal spindle-attachment region and a terminal satellite. It appears to be much like the satellited pair found in the fox and in many plants.[5] The other has a median spindle-attachment region and the satellite is connected at the same region as the spindle fibre (Fig. 1B). The chromosome itself is almost twice the length of its satellited mate. The similarity between these two chromosomes is striking if one considers the possibility of the smaller having arisen from the larger by the loss of an arm. Chromosomes with median attached satellites have been found in *Tradescantia* by Darlington[1] and in *Allium* by Levan,[3] and with laterally attached satellites in *Agrostis* by Tinney.[4] As a general rule, satellited chromosomes of both plants and animals are found associated with the nucleolus. This does not seem to hold true in the mink, at least not for the median satellited chromosome (Fig. 2). Neither Levan nor Tinney found any relationship between the chromosomes with median or laterally attached satellites and the nucleoli. Since the male is known to be the heterogametic sex in many mammals, the satellited pair of chromosomes may represent the sex chromosomes of the mink. Thus far, no satisfactory chromosome configuration from female tissues has been obtained.

The haploid number of chromosomes is more difficult to determine because the chromosomes are shorter, thicker and more closely associated on the meiotic equatorial plates than in the case of a mitotic metaphase. However, the 14 chromosomes on the equatorial plate of the second meiotic division confirms the diploid count herein reported (Fig. 3).

Spindle fibres and the resulting astral formations are clearly revealed by the aceto-carmine staining method (Fig. 4). The chromosomes in Figure 4 are approaching the equatorial plate for the first meiotic division, and chiasmata are to be seen in two of the chromosome pairs (Fig. 4C). No evidence of either precocious or lagging chromosomes was found in meiotic or mitotic anaphases (compare Figs. 5 and 6).

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